A Day of Recreational Mathematics (hosted by U Calgary)

Three reasons why recreational mathematics is engaging, fun, beautiful and important (and a triple dealing card trick)

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Richard Kenneth Guy (1916-2020) spent the second half of his life in Calgary. He loved working and playing—for him those were the same things—especially with young people. He loved geometry and numbers, and the Rockies; he climbed mountains well into his 90s.

His career was living proof that playful queries can both excite students about mathematics AND lead to real research at the frontiers of the subject. The message is clear:

Recreational math is engaging, fun, beautiful, and important.

First, a few words of interest (RKG loved word play too)

What's so special about

1. *Undercharge Thy Kin*

2. *Guy Rec Variations Fly = Guy Is Very Fractional*
   = Guy Left Visionary Arc

3. *Almost Wronged*

?

Hints: who are we talking about, where did he work and what did he really like other than mathematics?

Answers at the end.

Reason 1: Cake Cutting

Imagine an ideal rectangular cake with lime icing uniformly spread on the top and sides, but not on the bottom. It's easy to cut it with a knife so that 2 people get equal amounts of cake and equal amounts of icing.

Q1. How can this be achieved for 3 (or more) people, using only straight knife cuts?

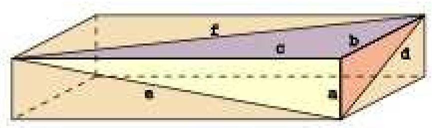
Assume we can cut $\frac{1}{3}$ of the way (or likewise) along any side, parallel to the perpendicular sides.

Reason 2: Perfect Packages

Consider traditional "Pythagorean" triples of whole numbers such as (3,4,5) or (5,12,13). The smaller numbers can be viewed as the lengths of the sides a rectangle whose (equal) diagonals—lengths given by the largest numbers—are also whole numbers.

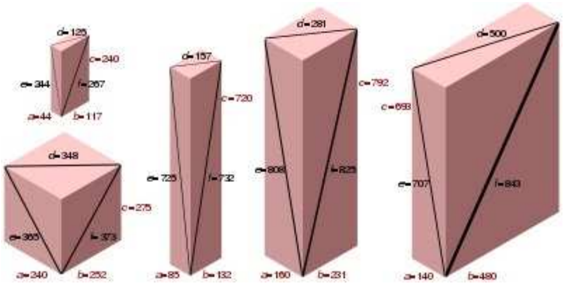
We can try to go up a dimension and ask:

> *Q2. Is there a 3D rectangular box all of whose sides and diagonals are whole numbers?*



Do we also insist that this includes the 4 equal space diagonals?

Without that requirement, a $44 \times 117 \times 240$ box works. So does $240 \times 252 \times 275$, along with the other "small" examples shown.



These are called Euler (not from Edmonton) bricks.

The bottom left box here is the most cubelike. No such actual cube can exist. (Why not?)

Is there a solution where the 4 equal space diagonals are whole numbers too? Do perfect Euler bricks exist?

Reason 3: Sums of Three Cubes

We've known since 1779 (Euler) that a sum of two cubes of whole numbers can't equal a cube of a whole number. Fermat's Last Theorem (1995) extends this from cubes to any n-th powers. How about sums of *three* cubes?

$$x^3 + y^3 + z^3 = w^3.$$

"Plato's number" $3^3 + 4^3 + 5^3 = 6^3$ is a noteworthy example. Ramanujan gave a formula that generates an infinite number of solutions (but does it give them all?).

Question: given n find non-zero x, y, z with $x^3 + y^3 + z^3 = n$. There is no hope if n is congruent to 4 or 5 mod 9. Or if $n = 0$. Subtraction is allowed, since the cube of a negative is negative.

What is the smallest (non-trivial!) solution for $n = 1$? Is it $10^3 + 9^3 + (-12)^3 = 1$? No! As well as such isolated solutions, there are infinite families of solutions.

What n work? When are there infinitely many solutions?

For $n = 2$ we have $1214928^3 + 3480205^3 + (-3528875)^3 = 2$ as the smallest solution.

There are two other isolated solutions and the following infinite family of solutions (1908, A.S. Verebrusov)

$$(1 + 6u^3)^3 + (1 - 6u^3)^3 + (-6u^2)^3 = 2.$$

In 1953, Mordell noted $1^3 + 1^3 + 1^3 = 4^3 + 4^3 + (-5)^3 = 3$ as the only known representation of 3 as a sum of three cubes.

No others were found—or ruled out—until very recently.

Forget 4 and 5.

Next, $2^3 + (-1)^3 + (-1)^3 = 6 = 65^3 + (-58)^3 + (-43)^3 = \dots.$

By 2016, all but two reasonable n up to 100 were known to work.

For 33, Andrew Booker (Mar 2019) found the 16-digit solutions

$$x = 8,866,128,975,287,528,$$

$$y = -8,778,405,442,862,239,$$

$$z = -2,736,111,468,807,040.$$

For 42, Andrew Booker & Andrew Sutherland (Sep 2019) found

$$x = 12,602,123,297,335,631,$$

$$y = 80,435,758,145,817,515,$$

$$z = -80,538,738,812,075,974.$$

(The ghost of Douglas Adams can now rest easy)

The next frontier: is 114 a sum of 3 cubes? Maybe yes, maybe no.

Curiously, if the answer is no, we might never know that, because there might be no way to prove it (see the 1970 resolution of Hilbert's Tenth Problem). There are about a half dozen other numbers under 1000 whose status is unknown.

Note: we know that every integer (including 4, 5, 13, 14, ...) is a sum of three cubes of *rational* numbers (1825, Leeds schoolmaster S. Ryley, *Ladies' Diary*). For instance, $(\frac{2}{3})^3 + (\frac{7}{3})^3 = 13$; this representation is related to rational points on an elliptic curve. (Compare this with Hilbert's Seventeenth Problem.)

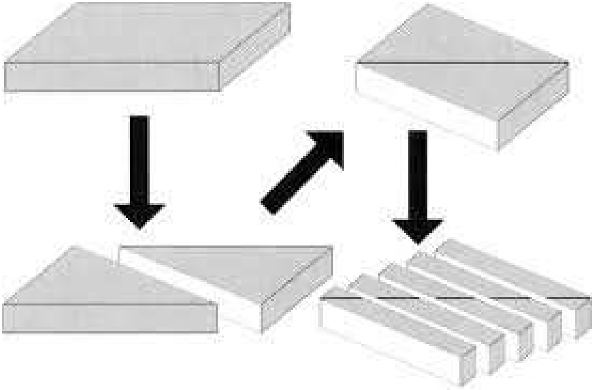
Back to 3: is it a sum of three cubes in any surprising ways?

In Sep 2019, Andrew Booker & Andrew Sutherland found:

$569936821221962380720^3 + (-569936821113563493509)^3 + (-472715493453327032)^3 = 3.$

So many questions! Here is one answer, for cake cutting:

Rectangular cake shared among 5 people



(Thinking outside the box? We basically turned the cake inside out!)

Complicated Cake Conundrums

A much harder version concerns cakes which are not regular in shape or uniform in composition, either inside or in their icing, and where the preferences of the participants (for cake, fruit, cream, icing, etc) may vary. The solution here for 2 people has been known since biblical times (chapter 13, *Book of Genesis*).

In the 1940s, Steinhaus figured out how to guarantee that each of n people gets a piece they value as worth at least $1/n$. It doesn't guarantee that they don't value someone else's piece more highly.

"Envy-free" cake cutting introduced in the 1950s (Gamow & Stern), and a solution for 3 people was developed around 1960 (Selfridge & Conway). That version extended (with difficulty) to 4 or 5 people.

In 2016, Aziz & Mackenzie had a breakthrough: envy-free cake-cutting can be done in bounded time for 4 or more people.

The life cycle

"This is the life cycle in recreational math: playfulness, 'Aha!' solutions & mathematical exploration, all leading back to new playful observations"

(Martin Gardner biographer and computer scientist Dana Richards)

A surprising mathematical card trick

Know the bottom card in a packet of size 12 or 13 or 14 (your *key* card). Pick your fave ice cream flavour. “Mango” is too short, “German chocolate chip cookie dough” is too long.

Deal cards to a pile on the table, one for each letter of “vanilla” or “strawberry” or “chocolate” or whatever, before dropping the rest on top. Do this two more times, spelling the same word each time.

This is a *reversed transfer* of a fixed number of cards (*at least half*) in a packet, from top to bottom, done three times total.

After the third round, the card on top is your key card.

WHY? (that’s the big question)

Look up “Low-Down Triple Dealing” or “The ice cream trick”.

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His career was living proof that playful queries can both excite students about mathematics AND lead to real research at the frontiers of the subject. I hope I have convinced you that

Recreational math is engaging, fun, beautiful, and important.

A few final words of interest:

What's so special about

1. *Undercharge Thy Kin?* It's an anagram of "Richard Kenneth Guy".

2. *Guy Rec Variations Fly = Guy Is Very Fractional = Guy Left Visionary Arc ?* They are all Richardian anagrams of "University of Calgary".

3. *Almost Wronged?* It's a *mountain phrase*—can you see why? (And it's also an anagram of "Downstream Log" and of "Montage Worlds".*)*